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Schur-convexity of the Catalan–Qi function related to the Catalan numbers

Feng Qi[1], Xiao-Ting Shi[2], Mansour Mahmoud[3] and Fang-Fang Liu[4]

[1]Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China; Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China
[2]Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China
[3]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt
[4]Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China
E-mail: qifeng618@gmail.com[1], xiao-ting.shi@hotmail.com[2], mansour@mans.edu.eg[3], fang-liu@qq.com[4]

Abstract

In the paper, the authors present the Schur-convexity of the absolute of the logarithm of the Catalan–Qi function and prove the logarithmically complete monotonicity of the Catalan–Qi function.

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1 Introduction

It is common knowledge in combinatorics [4, 7] that the Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" whose solution is the Catalan number C_{n-2}. The Catalan numbers C_n can be generated by

$$\frac{2}{1+\sqrt{1-4x}} = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n$$
$$= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + 1430x^8 + \cdots.$$

One of explicit formulas of C_n for $n \ge 0$ reads that

$$C_n = \frac{4^n \Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \tag{1.1}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d}\, t, \quad \Re(z) > 0$$

is the classical Euler gamma function.

In [28], among other things, a generalization of the expression (1.1) for the Catalan numbers C_n was given by

$$C(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)} \left(\frac{b}{a}\right)^z \frac{\Gamma(z+a)}{\Gamma(z+b)}, \quad \Re(a), \Re(b) > 0, \quad \Re(z) \ge 0, \tag{1.2}$$

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due to $C\left(\frac{1}{2}, 2; n\right) = C_n$. For uniqueness and convenience of referring to the quantity $C(a,b;x)$, we call the function $C(a,b;x)$ the Catalan–Qi function. It is clear that

$$C(a,b;0) = C(a,b;1) = 1 \tag{1.3}$$

and

$$C(a,b;x) = \frac{1}{C(b,a;x)}. \tag{1.4}$$

Currently we do not know and understand the combinatorial interpretation of $C(a,b;x)$. Here we would not like to discuss its combinatorial interpretation.

Recently, in the papers [8, 13, 19, 20, 24, 26, 27, 28, 33], the authors presented asymptotic expansions, integral representations, logarithmic convexity, complete monotonicity, minimality, logarithmically complete monotonicity, a generating function, and inequalities of the Catalan numbers C_n, the Catalan function C_x, the Catalan–Qi function $C(a,b;x)$ in x, and related functions involving the ratios $\frac{\Gamma(x+a)}{\Gamma(x+b)}$ and $\frac{\Gamma(a)}{\Gamma(b)}$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{R}^n$. A sequence λ is said to be majorized by μ (in symbols $\lambda \preceq \mu$) if $\sum_{\ell=1}^{k} \lambda_{[\ell]} \le \sum_{\ell=1}^{k} \mu_{[\ell]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{\ell=1}^{n} \lambda_\ell = \sum_{\ell=1}^{n} \mu_\ell$, where $\lambda_{[1]} \ge \lambda_{[2]} \ge \dots \ge \lambda_{[n]}$ and $\mu_{[1]} \ge \mu_{[2]} \ge \dots \ge \mu_{[n]}$ are respectively the components of λ and μ in decreasing order. A sequence λ is said to be strictly majorized by μ (in symbols $\lambda \prec \mu$) if λ is not a permutation of μ. For example,

$$\Bigg(\underbrace{\frac{1}{n}, \dots, \frac{1}{n}}_{n}\Bigg) \prec \Bigg(\underbrace{\frac{1}{n-1}, \dots, \frac{1}{n-1}}_{n-1}, 0\Bigg) \prec \Bigg(\underbrace{\frac{1}{n-2}, \dots, \frac{1}{n-2}}_{n-2}, 0, 0\Bigg) \prec \cdots \\ \prec \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \dots, 0\right) \prec \left(\frac{1}{2}, \frac{1}{2}, 0, \dots, 0\right) \prec (1, 0, \dots, 0).$$

Recall from [10, p. 80] and [11, pp. 75–76] that, a function f with n arguments defined on I^n is called Schur-convex if $f(x) \le f(y)$ for each two n-tuples $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ on I^n such that $x \prec y$ holds, where I is an interval with nonempty interior, and that a function f is Schur-concave if and only if $-f$ is Schur-convex.

Recall from [1, 17, 18] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if

$$0 \le (-1)^k [\ln f(x)]^{(k)} < \infty$$

hold on I for all $k \in \mathbb{N}$. For more information on logarithmically completely monotonic functions, please refer to [2, 21, 22, 23, 30, 32].

In this paper, we investigate the Schur-convexity of the Catalan–Qi function $C(a,b;x)$ in $(a,b) \in (0,\infty) \times (0,\infty)$ for all $x \ge 0$ and study the logarithmically complete monotonicity of $C(a,b;x)$ with respect to the variables $a > 0$ and $b > 0$ for all $x \ge 0$.

The main results of this paper are concluded in Theorems 1.1 and 1.2 below.

Theorem 1.1. *For $a, b > 0$ and $x \ge 0$, let*

$$F_x(a,b) = |\ln C(a,b;x)|. \tag{1.5}$$

Then the function $F_x(a,b)$ is Schur-convex in $(a,b) \in (0,\infty)\times(0,\infty)$ for all $x \ge 0$. In other words, if and only if $(a_1,b_1) \preceq (a_2,b_2)$, the inequality

$$|\ln C(a_1,b_1;x)| \le |\ln C(a_2,b_2;x)|$$

is valid for all $x \ge 0$.

Theorem 1.2. *Let $a,b>0$ and $x \ge 0$. Then the function $[C(a,b;x)]^{\pm 1}$ is logarithmically completely monotonic*

1. *with respect to $a>0$ if and only if $x \gtreqless 1$,*
2. *with respect to $b>0$ if and only if $x \lesseqgtr 1$.*

2 Lemmas

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 ([24, Theorem 4.2])**.** *Let $a,b>0$ and $x \ge 0$. Then*

1. *when $b>a$, the function $C(a,b;x)$ is decreasing in $x \in [0,x_0)$, increasing in $x \in (x_0,\infty)$, and logarithmically convex in $x \in [0,\infty)$;*
2. *when $b<a$, the function $C(a,b;x)$ is increasing in $x \in [0,x_0)$, decreasing in $x \in (x_0,\infty)$, and logarithmically concave in $x \in [0,\infty)$;*

where x_0 is the unique zero of the equation

$$\frac{\psi(x+b)-\psi(x+a)}{\ln b-\ln a}=1$$

and satisfies $x_0 \in \left(0,\frac{1}{2}\right)$.

Lemma 2.2 ([10, p. 84], [11, p. 333, Theorem 12.25], and [31, p. 259, Theorem C])**.** *Let $f(x)=f(x_1,\dots,x_n)$ be symmetric and have continuous partial derivatives on I^n, where I is an open interval. Then $f: I^n \to \mathbb{R}$ is Schur-convex if and only if*

$$(x_i-x_j)\left[\frac{\partial f(x_1,\dots,x_n)}{\partial x_i}-\frac{\partial f(x_1,\dots,x_n)}{\partial x_j}\right] \ge 0, \quad 1 \le i,j \le n \tag{1.6}$$

on I^n. The function f is strictly Schur-convex if the inequality (1.6) is strict for $x_i \ne x_j$.

Remark 2.1. By the way, the definition of the Schur-convexity and Lemma 2.2 were generalized and applied in [3, 12, 25, 34, 35, 37, 38, 39, 41] and closely-related references therein.

3 Proofs of Theorems 1.1 and 1.2

We are now in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. From the identity in (1.4), we deduce that the function (1.5) is symmetric with respect to a and b. By virtue of Lemma 2.1 and the identity (1.3), we see that

1. for $b > a > 0$,

$$F_x(a,b) = \begin{cases} -\ln C(a,b;x) = \ln C(b,a;x), & x \in [0,1], \\ \ln C(a,b;x) = -\ln C(b,a;x), & x \in [1,\infty); \end{cases}$$

2. for $a > b > 0$,

$$F_x(a,b) = \begin{cases} \ln C(a,b;x) = -\ln C(b,a;x), & x \in [0,1], \\ -\ln C(a,b;x) = \ln C(b,a;x), & x \in [1,\infty). \end{cases}$$

When $x \in [0,1]$ and $b > a > 0$, we have

$$\frac{\partial F_x(a,b)}{\partial a} = -\frac{\partial}{\partial a}\left\{\ln\left[\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^x \frac{\Gamma(x+a)}{\Gamma(x+b)}\right]\right\} = \psi(a) - \psi(x+a) + \frac{x}{a}, \tag{1.7}$$

$$\frac{\partial F_x(a,b)}{\partial b} = -\frac{\partial}{\partial b}\left\{\ln\left[\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^x \frac{\Gamma(x+a)}{\Gamma(x+b)}\right]\right\} = \psi(x+b) - \psi(b) - \frac{x}{b},$$

$$\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a} = [\psi(x+a) + \psi(x+b)] - [\psi(a) + \psi(b)] - x\left(\frac{1}{a} + \frac{1}{b}\right),$$

$$\frac{\partial}{\partial x}\left[\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a}\right] = \psi'(x+a) + \psi'(x+b) - \left(\frac{1}{a} + \frac{1}{b}\right),$$

$$\frac{\partial^2}{\partial x^2}\left[\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a}\right] = \psi''(x+a) + \psi''(x+b) < 0. \tag{1.8}$$

This means that the function

$$\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a} \tag{1.9}$$

is concave and

$$\lim_{x\to 0^+}\left[\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a}\right] = \lim_{x\to 1}\left[\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a}\right] = 0, \tag{1.10}$$

where we used the well-known identity $\psi(x+1) = \psi(x) + \frac{1}{x}$ on $(0,\infty)$ in the last step. Consequently, it follows that

$$(b-a)\left[\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a}\right] \ge 0 \tag{1.11}$$

for $b > a > 0$ and $x \in [0,1]$.

When $x \in [0,1]$ and $a > b > 0$, we have

$$\frac{\partial F_x(a,b)}{\partial a} = \frac{\partial}{\partial a}\left\{\ln\left[\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^x\frac{\Gamma(x+a)}{\Gamma(x+b)}\right]\right\} = \psi(x+a) - \psi(a) - \frac{x}{a}, \tag{1.12}$$

$$\frac{\partial F_x(a,b)}{\partial b} = \frac{\partial}{\partial b}\left\{\ln\left[\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^x\frac{\Gamma(x+a)}{\Gamma(x+b)}\right]\right\} = \psi(b) - \psi(x+b) + \frac{x}{b},$$

$$\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a} = [\psi(a)+\psi(b)] - [\psi(x+a)+\psi(x+b)] + x\left(\frac{1}{a}+\frac{1}{b}\right),$$

$$\frac{\partial}{\partial x}\left[\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a}\right] = \left(\frac{1}{a}+\frac{1}{b}\right) - [\psi'(x+a)+\psi'(x+b)],$$

$$\frac{\partial^2}{\partial x^2}\left[\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a}\right] = -[\psi''(x+a)+\psi''(x+b)] > 0. \tag{1.13}$$

This means that the function (1.9) is convex and the limits in (1.10) are still valid. Consequently, the inequality (1.11) still holds for $a > b > 0$ and $x \in [0,1]$.

When $x \in [1,\infty)$ and $b > a > 0$, the equations between (1.12) and (1.13) are still valid. This means that the function (1.9) is convex. The second limit in (1.10) is still valid and

$$\lim_{x\to\infty}\left[\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a}\right] = \infty,$$

where we used the limit

$$\frac{x}{a} - \psi(x+a) > \frac{x}{a} - \ln(x+a) + \frac{1}{2(x+a)} \to \infty, \quad a > 0, \quad x \to \infty,$$

which is deduced from the right-hand side of the double inequality

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}, \quad x > 0$$

collected in [5, p. 105, (1.5)]. An easy computation yields

$$\lim_{x\to 1}\frac{\partial}{\partial x}\left[\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a}\right] = \frac{1}{a} - \psi'(a+1) + \frac{1}{b} - \psi'(b+1) > 0,$$

where we used in the last step the inequality $\psi'(x+1) < \frac{1}{x}$ on $(0,\infty)$, which is a special case of [6, Theorem 3, (13)] and [14, p. 55, (5.17)] for $\beta = 1$ therein. Therefore, the function (1.9) is increasing on $[1,\infty)$, then the inequality (1.11) is valid for $b > a > 0$ and $x \in [1,\infty)$.

When $x \in [1,\infty)$ and $a > b > 0$, the equations between (1.7) and (1.8) are still valid. This means that the function (1.9) is concave. The second limit in (1.10) is still valid and

$$\lim_{x\to\infty}\left[\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a}\right] = -\infty.$$

An easy computation yields

$$\lim_{x\to 1}\frac{\partial}{\partial x}\left[\frac{\partial F_x(a,b)}{\partial b} - \frac{\partial F_x(a,b)}{\partial a}\right] = \psi'(a+1) - \frac{1}{a} + \psi'(b+1) - \frac{1}{b} < 0.$$

Therefore, the function (1.9) is decreasing on $[1,\infty)$, then the inequality (1.11) is valid for $a>b>0$ and $x\in[1,\infty)$.

In a word, by Lemma 2.2, we conclude that the function $F_x(a,b)$ for all $x\ge 0$ is Schur-convex in $(a,b)\in(0,\infty)\times(0,\infty)$. The proof of Theorem 1.1 is complete. Q.E.D.

Proof of Theorem 1.2. Taking the logarithm of $C(a,b;x)$, differentiating with respect to a, and applying the integral representation

$$\psi(z)=\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-zt}}{1-e^{-t}}\right)\mathrm{d}\,t,\quad \Re(z)>0$$

arrive at

$$\begin{aligned}\frac{\partial[\ln C(a,b;x)]}{\partial a}&=\frac{\partial}{\partial a}\ln\left[\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^x\frac{\Gamma(x+a)}{\Gamma(x+b)}\right]\\&=\psi(x+a)-\psi(a)-\frac{x}{a}\\&=\int_0^\infty\left(\frac{1-e^{-xt}}{1-e^{-t}}-x\right)e^{-at}\,\mathrm{d}\,t\\&=x\int_0^\infty\left(\frac{1-e^{-xt}}{xt}-\frac{1-e^{-t}}{t}\right)\frac{te^{-at}}{1-e^{-t}}\,\mathrm{d}\,t.\end{aligned}$$

Since the function $\frac{1-e^{-t}}{t}$ is strictly decreasing on $(0,\infty)$, then

$$\frac{1-e^{-xt}}{xt}-\frac{1-e^{-t}}{t}\gtrless 0$$

if and only if $x\lessgtr 1$. The Bernstein-Widder theorem, [36, p. 161, Theorem 12b], states that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $(0,\infty)$ is that

$$f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\,\mu(t),\tag{1.14}$$

where μ is a positive measure on $[0,\infty)$ such that the integral (1.14) converges on $(0,\infty)$. Consequently, the function $[C(a,b;x)]^{\pm1}$ is logarithmically completely monotonic with respect to $a>0$ if and only if $x\gtrless 1$.

By virtue of (1.2), we obtain immediately that the function $[C(a,b;x)]^{\pm1}$ is logarithmically completely monotonic with respect to $b>0$ if and only if $x\lessgtr 1$. The proof of Theorem 1.2 is complete. Q.E.D.

Remark 3.1. How about the Schur-harmonic convexity, Schur-geometric convexity, and Schur-m-power convexity of the Catalan–Qi function $C(a,b;x)$ in $(a,b)\in(0,\infty)\times(0,\infty)$ for all $x\ge 0$? For information on the Schur-harmonic convexity, Schur-geometric convexity, Schur-m-power convexity, and the like, please refer to [3, 35, 37, 38, 39, 40, 41] and closely-related references therein.

Remark 3.2. This paper is a companion of the articles [8, 9, 13, 15, 16, 19, 24, 26, 27, 28, 33] and a slightly revised version of the preprint [29].

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